Corrigé du contrôle TD 2605 20/10/2021 ①

Question de cours — voir wikipedia, cours de M. Lafay etc.

Exercice 1

$$A = \begin{pmatrix} 2 & -3 & -1 \\ 1 & -1 & -2 \\ -3 & -2 & 3 \end{pmatrix}$$

1) 1 matrice carrée A est diagonalisable ssi $\exists$ 2 matrices P et D (P inversible et D matrice diagonale) telles que

$$A = PDP^{-1}$$

2)

$$p(\lambda) = \begin{vmatrix} 2-\lambda & -3 & -1 \\ 1 & -1-\lambda & -2 \\ -3 & -2 & 3-\lambda \end{vmatrix} \overset{c_1+c_2+c_3}{=} \begin{vmatrix} -2-\lambda & -3 & -1 \\ -2-\lambda & -1-\lambda & -2 \\ -2-\lambda & -2 & 3-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} -2-\lambda & -3 & -1 \\ 0 & 2-\lambda & -1 \\ 0 & 1 & 4-\lambda \end{vmatrix} \begin{matrix} \\ L_2 - L_1 \\ L_3 - L_1 \end{matrix}$$

$$= (-2-\lambda) \begin{vmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{vmatrix} = (2-\lambda)\left[(2-\lambda)(4-\lambda)+1\right]$$

$$= (2-\lambda)\left[\lambda^2 - 6\lambda + 9\right]$$

$\Delta = (-6)^2 - 4 \times 9 = 0$

$\Rightarrow$ 1 racine double $\lambda_1 = \lambda_2 = \frac{-b}{2a} = 3$

$\Rightarrow p(\lambda) = (-2-\lambda)(3-\lambda)^2$

$$\boxed{\Rightarrow \underline{vp} : \begin{array}{ll} \lambda_1 = \lambda_2 = 3 & \text{vp double} \\ \lambda_3 = -2 & \text{vp simple.} \end{array}}$$

3) Non on ne peut rien dire. cf corrigé contrôle $\hookrightarrow$ 6C 0/

4) $\underline{E_{\lambda_1=\lambda_2=3}}$: $(A - 3I)\vec{u} = \vec{0}$

$$\begin{cases} -x - 3y - z = 0 \\ x - 4y - 2z = 0 \\ -3x - 2y = 0 \end{cases} \Leftrightarrow \begin{cases} -x - 3y - z = 0 & \\ -7y - 3z = 0 & L_2 + L_1 \\ 7y + 3z = 0 & L_3 - 3L_1 \end{cases}$$

$$\Leftrightarrow \begin{cases} -x - 3y - z = 0 & \\ -7y - 3z = 0 & \\ 0 = 0 & L_3 + L_2 \end{cases} \Leftrightarrow \begin{cases} x = -3y - z = \left(+\frac{9}{7} - 1\right) z \\ y = -3/7\, z \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 2/7\, z \\ y = -3/7\, z \\ z = z \end{cases}$$

$$E_{\lambda_1=\lambda_2=3} = \left\{ z \begin{pmatrix} 1 \\ -3/2 \\ 7/2 \end{pmatrix} ; z \in \mathbb{R} \right\}.$$

base $\left\{ \begin{pmatrix} 1 \\ -3/2 \\ 7/2 \end{pmatrix} \right\}$ $\dim E_{\lambda_1=\lambda_2} = 1$

$\underline{E_{\lambda_3} = -2}$? $(A+2I)\vec{u} = \vec{0}$

$$\begin{cases} 4x - 3y - z = 0 \\ x + y - 2z = 0 \\ -3x - 2y + 5z = 0 \end{cases} \Leftrightarrow \begin{cases} x + y - 2z = 0 & L'_1 = L_2 \\ 4x - 3y - z = 0 & L'_2 = L_1 \\ -3x - 2y + 5z = 0 \end{cases}$$

$$\begin{cases} x + y - 2z = 0 \\ -7y + 7z = 0 & L_2 - 4L_1 \\ y - z = 0 & L_3 + 3L_1 \end{cases} \Leftrightarrow \begin{cases} x + y - 2z = 0 \\ -y + z = 0 \\ y - z = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = -y + 2z \\ y = z \\ 0 = 0 & L_3 + L_2 \end{cases} \Leftrightarrow \begin{cases} x = z \\ y = z \\ z = z \end{cases}$$

$$\boxed{E_{\lambda_3 = -2} = \left\{ z \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} ; z \in \mathbb{R} \right\} \quad \text{base} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \quad \dim = 1}$$

5) là on peut dire que - Matrice pas diagonalisable. Voir corrigé TD GC 01.

Exercice 2

$$A = \begin{pmatrix} 2 & -1 & 2 \\ 1 & 2 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

1) $$\det A = \begin{vmatrix} 2 & -1 & 2 \\ 1 & 2 & -1 \\ -1 & 0 & 1 \end{vmatrix} = -1 \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}$$

$$= -(1-4) + 5 = 3 + 5 = 8 \neq 0 \Rightarrow A \text{ inversible.}$$

2) $$\mathrm{Co}\,A = \begin{pmatrix} +\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ -1 & 0 \end{vmatrix} \\ +\begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} \end{pmatrix}$$

$$\mathrm{Co}\,A = \begin{pmatrix} 2 & 0 & 2 \\ 1 & 4 & 1 \\ -3 & 4 & 5 \end{pmatrix} \Rightarrow {}^t\mathrm{Co}\,A = \begin{pmatrix} 2 & 1 & -3 \\ 0 & 4 & 4 \\ 2 & 1 & 5 \end{pmatrix}$$

finalement
$$\boxed{A^{-1} = \frac{1}{8}\begin{pmatrix} 2 & 1 & -3 \\ 0 & 4 & 4 \\ 2 & 1 & 5 \end{pmatrix}}$$

verif rapide
$$A^{-1}A = \frac{1}{8}\begin{pmatrix} 2 & 1 & -3 \\ 0 & 4 & 4 \\ 2 & 1 & 5 \end{pmatrix}\begin{pmatrix} 2 & -1 & 2 \\ 1 & 2 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{8}\begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ Ok

Exercice 3

1) $f : E \to F \qquad E = \mathbb{R}^3 \qquad F = \mathbb{R}^3$

2) cf corrigé TD GC01

3) $\ker f = \{ \vec{u} \in E \,;\, f(\vec{u}) = \vec{0}_F \}$

4) Ker f ?

$$(S)\begin{cases} x + 3\alpha y - 3z = 0 \\ -3x + (3-9\alpha)y + 11z = 0 \\ 2x + (-6+6\alpha)y + (3\alpha - 11)z = 0 \end{cases} \Leftrightarrow \begin{cases} x + 3\alpha y - 3z = 0 \\ 3y + 2z = 0 : L_2 + 3L_1 \\ -6y + (3\alpha - 5)z = 0 : L_3 - 2L_1 \end{cases}$$

$$\Leftrightarrow \begin{cases} x + 3\alpha y - 3z = 0 \\ 3y + 2z = 0 \\ (3\alpha - 1)z = 0 \quad : L_3 + 2L_1 \end{cases}$$

1er cas : si $3\alpha - 1 \neq 0 \Leftrightarrow \alpha \neq 1/3$ on peut diviser

$$(S) \Leftrightarrow \begin{cases} x + 3\alpha y = 0 \\ 3y = 0 \\ z = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$$

$\Rightarrow$ Si $\alpha \neq 1/3$

$\ker f_\alpha = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$; pas de base

$\dim \ker f_\alpha = 0$

<u>2e cas</u> : si $\alpha = 1/3$, on remplace α dans le système.

$$(S) \iff \begin{cases} x + y - 3z = 0 \\ 3y + 2z = 0 \\ 0 = 0 \end{cases} \iff \begin{cases} x = -y + 3z \\ y = -2/3\, z \end{cases}$$

$$\iff \begin{cases} x = (2/3 + 3)z = 11/3\, z \\ y = -2/3\, z \\ z = z \end{cases}$$

- si $\alpha = 1/3$
$\mathrm{Ker} f = \left\{ z \begin{pmatrix} 11/3 \\ -2/3 \\ 1 \end{pmatrix} ; z \in \mathbb{R} \right\}$
base $\left\{ \begin{pmatrix} 11/3 \\ -2/3 \\ 1 \end{pmatrix} \right\}$, $\dim \mathrm{Ker} f = 1$

verif rapide en reprenant syst. départ et en mettant $\alpha = 1/3$ avec $x = 11/3$, $y = -2/3$, $z = 1$

$$\begin{cases} x + y - 3z = 11/3 - 2/3 - 3 = 9/3 - 3 = 0 & \text{ok ✓} \\ -3x + 0y + 11z = -11 + 11 = 0 & \text{ok ✓} \\ 2x + (-6+2)y + (1-11)z = 22/3 - 4\left(-\frac{2}{3}\right) - 10 = 0 & \text{ok} \end{cases}$$

5) $\dim E = \mathrm{rg} f + \dim \mathrm{Ker} f \Rightarrow \mathrm{rg} f = 3 - \dim \mathrm{Ker} f$

<u>si $\alpha \neq 1/3$</u> : $\mathrm{rg} f = 3 - 0 = 3$

<u>si $\alpha = 1/3$</u> : $\mathrm{rg} f = 3 - 1 = 2$

6) On cherche d'abord 1 famille génératrice de $\mathrm{Im}\, f$ en prenant l'image de la base canonique.

(7)

$$\begin{cases} f(e_1) = f(1,0,0) = (1,-3,2) = v_1 \\ f(e_2) = f(0,1,0) = (3\alpha, 3-9\alpha, -6+6\alpha) = v_2 \\ f(e_3) = f(0,0,1) = (-3, 11, 3\alpha-11) = v_3 \end{cases}$$

$\mathrm{Im}\, f = \mathrm{Vect}\{v_1, v_2, v_3)$

<u>si $\alpha \neq 1/3$</u>, $\mathrm{rg}\, f = 3 \Rightarrow$ car 3 vecteurs sont libres

$\Rightarrow$ base de $\mathrm{Im}\, f = \{v_1, v_2, v_3\}$

<u>si $\alpha = 1/3$</u>, $\mathrm{rg}\, f = 2$ seuls 2 vecteurs sont libres, les 2 premiers ne sont pas proportionnels en effet

$$v_1 = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} \quad v_3 = \begin{pmatrix} -3 \\ 11 \\ -10 \end{pmatrix}$$

$\Rightarrow$ base de $\mathrm{Im}\, f = \{v_1, v_2\}$.

7) <u>si $\alpha \neq 1/3$</u> : . $\dim \ker f = 0 \Rightarrow f$ injective

. $\mathrm{rg}\, f = 3 = \dim F \Rightarrow f$ surjective

. f inj et surj $\Rightarrow f$ bijective

Si $\alpha = 1/3$:

$\begin{cases} \cdot \dim \ker f = 1 \neq 0 \Rightarrow f \text{ pas injective} \\ \cdot \operatorname{rg} f = 2 \neq 3 = \dim F \Rightarrow f \text{ pas surjective} \\ \cdot f \text{ ni injective ni surjective} \Rightarrow f \text{ pas bijective} \end{cases}$

8) $$M(f_\alpha) = \begin{array}{ccc} f(e_1) & f(e_2) & f(e_3) \\ \end{array}$$
$$M(f_\alpha) = \begin{pmatrix} 1 & 3\alpha & -3 \\ -3 & 3-9\alpha & 11 \\ 2 & -6+6\alpha & 3\alpha-11 \end{pmatrix}$$
$v_1 \quad v_2 \quad v_3$

par lecture de la fonction.

9) $M(f_\alpha)$ inversible $\Leftrightarrow f_\alpha$ bijective.
donc $M(f_\alpha)$ inversible ssi $\alpha \neq 1/3$

Exercice 4 Calculez la dérivée de

$$f(x) = (3x^2 + 2x)e^{2x}$$ de la forme $(uv)' = u'v + uv'$

$$f'(x) = (6x+2)e^{2x} + (3x^2+2x)\,2e^{2x}$$
$$= e^{2x}(6x+2+6x^2+4x)$$
$$\boxed{f'(x) = (6x^2+10x+2)\,e^{2x}}$$